THE ALLEN-CAHN EQUATION AND MINIMAL SURFACES (LSGNT, 2022)

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Abstract. The following is a list of problems with substantial hints and brief discussions, related to the topics of my talk at LSGNT on January 2022. The focus is on elementary properties of solutions both in 1D and higher dimensions, as well as min-max and other methods for constructing solutions.

1. Notation for hypersurfaces

Definition 1.1. Given $\Sigma \subset M$, we denote by $|\Sigma|:= \mathcal{H}^{n-1}(\Sigma)$ the $(n-1)$-dimensional Hausdorff measure of the set Σ, i.e. $|\Sigma|= \text{Area}(\Sigma)$, for $\Sigma = \Sigma^{n-1}$ a smooth embedded hypersurface.

Definition 1.2. Let $\Sigma \subset M$ be an embedded smooth hypersurface. We say that Σ is *separating* iff $M \setminus \Sigma$ is the union of two open regions M_1 and M_2 (not necessarily connected) and such that $\Sigma = \partial M_1 = \partial M_2$.

Example 1.3. The union of k disjoint slices (copies of $\mathbb{T}^{n-1}$) of a torus $\mathbb{T}^n$ form a separating hypersurface iff k is even. $\mathbb{RP}^2 \subset \mathbb{RP}^3$ is not separating. Indeed, since we are assuming M is orientable, then Σ cannot be separating whenever one of its connected components is non-orientable.

Definition 1.4. We say that an embedded minimal hypersurface Σ_0 (possibly unorientable) is *non-degenerate* iff $\frac{d^2}{dt^2}\text{Area}(\Sigma_t)|_{t=0}\neq 0$, for all smooth variations Σ_0, such that $\frac{d}{dt}\Sigma_t|_{t=0}$ is normal to Σ_t and not identically zero. If in addition, $\frac{d^2}{dt^2}\text{Area}(\Sigma_t)|_{t=0}\geq 0$ for every variation, we say that Σ is *stable* (or *strictly stable* if the inequality is strict).

Remark. Let Σ be an orientable minimal hypersurface and ∂ a continuous choice of a normal vector. If $\frac{d}{dt}\Sigma_t|_{t=0}= f\partial$, for $f \in C^\infty(\Sigma)$, then $\frac{d^2}{dt^2}\text{Area}(\Sigma_t)|_{t=0}= -\int_M fJ(f)$, where $J = \Delta_\Sigma + |A_0|^2 + \text{Ric}_M(\partial,\partial)$ is the Jacobi operator of Σ and A_0 is the second fundamental form of Σ (see [4]). In this case, we denote by $\text{Ind}(\Sigma)$ the number of negative eigenvalues of $-J$ (counted with multiplicity).

2. The Allen-Cahn equation

Let $\Omega \subset M^n$ be a region with smooth boundary, of an n-dimensional Riemannian manifold). Given $\varepsilon > 0$, we refer to the semilinear elliptic equation

$$\varepsilon^2\Delta u - W'(u) = 0, \tag{1}$$

where $u : \Omega \to \mathbb{R}$ is a $C^2(\Omega)$ function and $W(u) = (1-u^2)^2/4$, as the Allen-Cahn equation.

Notice that the constants ± 1 and 0 satisfy (1). We refer to these as *trivial* solutions of (1). A simple computation shows that (1) is the Euler-Lagrange equation of the energy

$$E_\varepsilon(u) = \int_M \varepsilon \frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon} \tag{2}$$

as a functional on the Sobolev space $W^{1,2}(M)$, i.e. u solves (1) iff it is a critical point of E_ε.

Definition 2.1. If u solves (1), we denote by $\mathrm{Ind}(u)$ the number of negative eigenvalues of the linearized operator $-L_u$ where $L_u = \varepsilon^2 \Delta - W''(u)$. Notice that $E''_\varepsilon(u)(\phi, \phi) = -\int_M \phi L_u \phi$.

Exercise 1 (ε-scaling)**.** *If u satisfies* (1) *on (M, g), then u satisfies* (1) *with $\varepsilon = 1$ on the rescaled metric $(M, \varepsilon^{-2} g)$.*

Exercise 2 (± 1 are the solution with lowest energy)**.** *Show that ± 1 are the only global minimizers and $E_\varepsilon(\pm 1) = 0$.*

Exercise 3 (0 is the solution with highest energy)**.** *Show that if u is any non zero solution of* (1) *(not necessarily trivial) then $E_\varepsilon(u) < E_\varepsilon(0)$.*

Hint: integrate by parts in (2)*, substitute $\varepsilon^2 \Delta u = u^3 - u$ and simplify.*

3. One-dimensional solutions

In this section we study the ODE $u'' - W'(u) = 0$, for $u : I \to \mathbb{R}$ where $I \subset \mathbb{R}$ is an open interval and $W'(u) = u^3 - u$. Solutions to this autonomous semilinear equation, can be classified (up to translations and reflections) in terms of their discrepancy $\xi = (u')^2/2 - W(u)$, where $W(u) = (1-u^2)^2/4$.

Exercise 4. $\frac{d}{dt}\xi(t) = 0$*, for $t \in I$. In other words, ξ is constant. On a phase diagram $u \times u'$, draw the curves corresponding to solutions of* (1) *for different values of ξ.*

3.1. **Strictly Monotone Solutions.** In a region where a solution is monotone we can express its inverse with an explicit integral. To derive the formula assume that u is a strictly monotone increasing solution on an interval I (so $u' > 0$ in I) and that $0 \in I$. Let $v : u(I) \to I$ be the inverse of u, i.e. $u \circ v = \mathrm{Id}_{u(I)}$ and $v \circ u = \mathrm{Id}_I$. Notice that $v'(s) = (u'(v(s)))^{-1}$ and from the discrepancy formula above $u'(v(s)) = \sqrt{2}|\xi + W(s)|^{1/2}$. Integrating, one gets

$$v(s) = (\sqrt{2}/2) \int_{u(0)}^{s} \frac{dr}{|\xi + W(r)|^{1/2}},$$

for all $s \in u(I)$.

In general, we can use the integral above to define solutions. Given any constants $u(0)$ and ξ, the integral above has a maximum interval of definition J which depends on the roots of the polynomial $\xi + W(r)$. In particular, $v(J)$ covers the maximum interval of definition of a solution with initial condition $u(0)$ and $u'(0) = \sqrt{2}|\xi + W(u(0))|^{1/2}$.

Exercise 5. *Show that if u is an entire solution with $u' > 0$, then $u(s) = 0$ for some $s \in J$ and $\xi = 0$.*

Since the equation is autonomous, we can translate the solution from the previous exercise it in order to assume $u(0) = 0$. From the exercise and the formulas above, it follows

$$\begin{aligned} v(s) &= (\sqrt{2}/2) \int_0^s \frac{dr}{|W(r)|^{1/2}} \\ &= \sqrt{2} \int_0^s \frac{dr}{1 - r^2} \\ &= \sqrt{2}\,\mathrm{arctanh}(s). \end{aligned} \tag{3}$$

Since v is the inverse of u, we conclude that $u(t) = \tanh(t/\sqrt{2})$. We just proved:

Exercise 6. *The only entire strictly monotone solution (up to translations and reflections) is given explicitly by the formula*

$$u(t) = \tanh(t/\sqrt{2}).$$

3.2. **Entire solutions with finite energy.** If $u : \mathbb{R} \to \mathbb{R}$ has finite energy $E(u) = \int_{\mathbb{R}} (u')^2/2 + W(u) < +\infty$, it follows that $\int_{\mathbb{R}} (u')^2/2$ and $\int_{\mathbb{R}} W(u)$ are both finite. From the definition of discrepancy $E(u) = \int_{\mathbb{R}} \xi + 2W(u) < +\infty$. Therefore, $\int_{\mathbb{R}} \xi < +\infty$. Since ξ is constant, it must vanish, i.e. $\xi = 0$.

If $u'(t_0) = 0$ for some t_0, then $0 = \xi = u'(t_0)^2/2 + W(u(t_0)) = W(u(t_0))$. We conclude $u(t_0) = \pm 1$ and $u'(t_0) = 0$. By uniqueness of ODE we conclude that $u(t) = \pm 1$, for all $t \in \mathbb{R}$.

If u' is never zero, without loss of generality we can assume $u' > 0$. This is the case studied in the previous subsection, where we concluded that (up to translations)

$$u(t) = \tanh(t/\sqrt{2}).$$

This shows

Exercise 7. *The only entire solutions with finite energy (up to translations and reflections) are the constants ± 1 and $u(t) = \tanh(t/\sqrt{2})$.*

3.3. **Other entire solutions.**

Exercise 8. *Besides the finite energy solutions discussed above, the only other entire solutions are periodic with initial conditions $u(0) \in (-1, 1)$ and $u'(0) = 0$ (up to translations). All of these solutions have finite energy. Notice this includes the zero solution, i.e. $u(t) = 0$, for all $t \in \mathbb{R}$.*

3.4. **Solutions that blow up in finite time.** All solutions to this equation, including the ones that blow up in finite time, can be studied in the same fashion as before using the integral formula. For completeness, the reader might want to compare what happens in those cases using the phase diagram. However, in practice finite energy solutions are the most relevant.

3.5. **Stability of the fundamental solution.**

Definition 3.1. From now on we refer to $\psi(t) = \tanh(t/\sqrt{2})$ as the fundamental solution.

The theory of regularity for solutions to the Allen-Cahn equation is based, among other things, on the fact that we can describe the stability properties of the fundamental solution, i.e. the kernel of the linearized ODE: $v'' - W''(\psi)v$.

Lemma 3.2. *Let $\ell(v) = v'' - W''(\psi)v$.*

(1) *ℓ is stable, i.e. for all $f \in W^{1,2}(\mathbb{R})$ we have $-\int_{\mathbb{R}} f\ell(f) \geq 0$.*
(2) *Equality holds only for $f(t) = \psi'(t)$, i.e. $\ker(\ell) = \operatorname{span}\langle\psi'\rangle$.*
(3) *There exists $\gamma > 0$, such that for all f orthogonal to the kernel, i.e. such that $\int_{\mathbb{R}} f\psi' = 0$, we have the estimate*
$$-\int_{\mathbb{R}} f\ell(f) \geq \gamma \int_{\mathbb{R}} f^2.$$

Proof. Proof of 1. Define ρ as $f = \rho\psi'$. Then

$$\begin{aligned}
-\int_{\mathbb{R}} f\ell(f) &= -\int_{\mathbb{R}} \rho\psi'\ell(\rho\psi') \\
&= \int_{\mathbb{R}} \rho\psi'[-(\rho\psi')'' + W''(\psi)\rho\psi'] \\
&= \int_{\mathbb{R}} -\rho\psi'(\rho\psi')'' + W''(\psi)(\rho\psi')^2 \\
&= \int_{\mathbb{R}} -\rho\psi'(\rho'\psi' + \rho\psi'')' + W''(\psi)(\rho\psi')^2 \\
&= \int_{\mathbb{R}} -\rho\psi'[\rho''\psi' + \rho'\psi'' + \rho'\psi'' + \rho(\psi')''] + W''(\psi)(\rho\psi')^2 \\
&= \int_{\mathbb{R}} -[\rho\rho''(\psi')^2 + (\rho^2)'([\psi'/2]^2)'] \\
&= \int_{\mathbb{R}} -\rho\rho''(\psi')^2 + (\rho')^2[\psi']^2 + \rho\rho''[\psi']^2 \\
&= \int_{\mathbb{R}} (\rho')^2[\psi']^2
\end{aligned}$$

For the proof of 2 and 3, we refer the reader to [11]. □

4. Higher dimensional solutions

4.1. **First Variation Formula for vector fields.** The first derivative of E_ε at a function u, is giving by the linear functional $v \mapsto E'_\varepsilon(u)(v) = \int_M \varepsilon\nabla u \cdot \nabla v + \frac{W'(u)}{\varepsilon}v$. A critical point u is then characterized by the condition
$$\int_M \varepsilon\nabla u \cdot \nabla v + \frac{W'(u)}{\varepsilon}v = 0, \quad \forall v \in W^{1,2}(M).$$

We could also restrict ourselves to understand the first variation of the energy only with respect to ambient vector fields. Let X be an ambient vector field

M and $\psi_t : M \to M$ be the flow generated by X, i.e. $\frac{d}{dt}\psi_t|_{t=0}= X$. In this case the infinitesimal variation of u at time zero, is given by $du(X) = \langle \nabla u, X\rangle$. Making $v = \langle \nabla u, X\rangle$ in the formula above and integrating by parts we obtain

Lemma 4.1 (First variation formula for vector fields)**.** *If u solves* (1)*, then for every smooth ambient vector field X we have*

$$\int_M \operatorname{div} X \cdot \left(\frac{\varepsilon|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}\right) = \int_M \langle \nabla_{\nabla u} X, \nabla u\rangle \tag{4}$$

4.2. Discrepancy.

Remark. Notice $\nabla|\nabla u|= \nabla_{\nabla u}\nabla u$. In fact,

$$\nabla_X|\nabla u|^2/2 = \langle \nabla_X \nabla u, \nabla u\rangle = \langle \nabla_{\nabla u}\nabla u, X\rangle.$$

On the other hand

$$\nabla_X|\nabla u|^2/2 = |\nabla u|\nabla_X|\nabla u|= |\nabla u|\langle \nabla|\nabla u|, X\rangle.$$

Exercise 9. *Show* $|\operatorname{Hess} u|^2-|\nabla|\nabla u||^2\geq 0$ *at every point.*

Lemma 4.2 (Bound on the discrepancy)**.** *Let M^n be a* ***closed*** *Riemannian manifold with* $\mathrm{Ric_M} > 0$ *and* $u : M \to \mathbb{R}$ *a solution to* (1)*. Then the discrepancy ξ satisfies* $\sup_M \xi \leq 0$ *and* $\sup_M \xi = 0$ *if and only if* $u \equiv \pm 1$*.*

Proof. Let $p \in M$ such that $\xi(p) = \sup_M \xi$. At this point we must have $\nabla\xi(p) = 0$ and $\Delta\xi(p) \leq 0$. By Remark 4.2 the first equation gives us:

$$\begin{aligned} 0 = \nabla\xi(p) &= \nabla\frac{|\nabla u(p)|^2}{2} + \nabla W(u(p)) \\ &= |\nabla u(p)|\cdot\nabla|\nabla u(p)|+W'(u(p))\nabla u(p) \end{aligned} \tag{5}$$

In particular, $|W'(u(p))|= |\nabla|\nabla u(p)||$ On the other hand, by Bochner's formula the second equation reads

$$\begin{aligned} 0 &\geq \Delta\xi(p) \\ &= |\operatorname{Hess} u|^2+W''(u)|\nabla u|^2+Ric_M(\nabla u, \nabla u) - W'(u)^2 - W''(u)|\nabla u|^2 \\ &= |\operatorname{Hess} u|^2-|\nabla|\nabla u(p)||^2+Ric_M(\nabla u, \nabla u) \\ &\geq Ric_M(\nabla u(p), \nabla u(p)). \end{aligned} \tag{6}$$

Since $Ric_M > 0$ this implies that $\nabla u(p) = 0$. Now, if $\xi(p) > 0$, then $|\nabla u(p)|^2/2 > W(u(p)) \geq 0$. In particular $|\nabla u(p)|\neq 0$ and this is a contradiction. If $\xi(p) = 0$, then $\nabla u(p) = 0$ implies $W(u(p)) = 0$, therefore $u(p) = \pm 1$ and by the maximum principle $u \equiv \pm 1$. □

In general, one cannot expect $\xi \leq 0$. However, there are some situations where a similar bound hold. When $M = \mathbb{R}^n$ entire solutions always satisfy $\xi \leq 0$ and equality holds only for the canonical 1-D solution $u(x) = \psi(x \cdot v)$, where v is a fixed non zero vector, and the constants ± 1. Similar to what happens with the bound for the supremum of u, one can recover the result in the limit, in a precise way, for solutions with bounded energy as $\varepsilon \to 0$.

4.3. **Second Variation Formula.** Remember Bochner's formula

$$\Delta|\nabla u|^2 = 2|\mathrm{Hess}(u)|^2 + 2\nabla\Delta u\cdot\nabla u + 2\,\mathrm{Ric}(\nabla u,\nabla u)$$

The following integrals are over the region $|\nabla u| \neq 0$ $f = g|\nabla u|$ and using Bochner's formula:

$$\begin{aligned}
\frac{d^2}{dt^2}\bigg|_{t=0} E(u+tf) &= E''(u)(f,f)\\
&= \int |\nabla f|^2 + W''(u)f^2\\
&= \int |\nabla(g|\nabla u|)|^2 + W''(u)(g|\nabla u|)^2\\
&= \int ||\nabla u|\nabla g + g\nabla|\nabla u||^2 + [\Delta\frac{|\nabla u|^2}{2} - |\mathrm{Hess}(u)|^2 - \mathrm{Ric}(\nabla u,\nabla u)]g^2\\
&= \int |\nabla u|^2|\nabla g|^2 + g^2|\nabla|\nabla u||^2 + 2g|\nabla u|(\nabla g\cdot\nabla|\nabla u|) - g\nabla|\nabla u|^2\cdot\nabla g\\
&\quad - (|\mathrm{Hess}(u)|^2 + \mathrm{Ric}(\nabla u,\nabla u))g^2\\
&= \int |\nabla u|^2|\nabla g|^2 - (|\mathrm{Hess}(u)|^2 - |\nabla|\nabla u||^2 + \mathrm{Ric}(\nabla u,\nabla u))g^2
\end{aligned}$$

Exercise 10. *Show that if* $\mathrm{Ric} > 0$ *then there are no stable solutions. Hint: try* $g =$ *constant on the second variation formula.*

5. Construction of solutions

Lemma 5.1 (Palais-Smale condition)**.** *Let* M *be a closed Riemannian manifold. Consider the functional* $E(u) = \int_M \frac{|\nabla u|^2}{2} + W(u)$. *Then,* E *satisfies the following Palais-Smale type condition a long sequences bounded in* L^∞*:*

Let $u_k \in W^{1,2}(M)$ *be a sequence such that:*

(1) $\sup_k \|u_k\|_{L^\infty(M)} < +\infty$
(2) $\sup_k E(u_k) < +\infty$
(3) $\|E'(u_k)\| = \sup\{|E'(u_k)(\phi)| : \|\phi\|_{W^{1,2}(M)} = 1\} \to 0.$

Then, there exists $u \in W^{1,2}(M)$ *such that* $u_k \to u$ *strongly in* $W^{1,2}(M)$ *(after perhaps passing to a subsequence). In particular,* u *is a solution to* (1).

Proof. First, notice that (1) and (2) imply that $\sup_k \|u_k\|_{W^{1,2}(M)} < +\infty$, since $\int_M |\nabla u|^2 \leq 2E(u)$. By Rellich-Kondrachov's compactness theorem, after passing to a subsequence we can assume there is $u \in W^{1,2}(M)$ such that $u_k \to u$ weakly in $W^{1,2}(M)$ and strongly in $L^2(M)$.

To complete the statement, we only need to show that $\int_M |\nabla(u_k - u)|^2 \to 0$. In the equality

$$\int_M |\nabla(u_k - u)|^2 = E'(u_k)(u_k - u) - \int_M \nabla u \cdot \nabla u_k + \int_M |\nabla u|^2 - \int_M W'(u_k)(u_k - u)$$

the first term goes to zero by (3) and $\sup_k \|u_k - u\|_{W^{1,2}(M)} < +\infty$. Second, $\int_M \nabla u \cdot \nabla u_k \to \int_M |\nabla u|^2$ since $u_k \to u$ weakly in $W^{1,2}(M)$ and strongly in $L^2(M)$. Finally, Hölder's inequality and (1) gives us

$$\left| \int_M W'(u_k)(u_k - u) \right| \leq \sup_{|s| \leq \|u_k\|_{L^\infty(M)}} |W'(s)| \cdot \mathrm{Vol}(M)^{1/2} \cdot \|u_k - u\|_{L^2(M)} \to 0.$$

□

Exercise 11. *Use the classical Mountain-Pass theorem (for example from Ambrosetti-Rabinowitz) to show that there are always mountain pass solutions to the equation above in closed manifolds.*

6. Other fundamental properties of solutions

Let M be a closed manifold and u a solution to (1) on M.

Exercise 12. *Show that $|u| \leq 1$. If the equality holds at one point then $u = \pm 1$.*

Hint: use the maximum principle.

Exercise 13. *If u is a non-trivial solution then $\{u = 0\} \neq \emptyset$.*

Hint: Assume $\{u = 0\} = \emptyset$ and try the linear deformation connecting u with $\mathrm{sgn}(u)$, *i.e* $(1 - t)u + t\mathrm{sgn}(u)$.

Exercise 14 (*)**.** *Show that if $u(p) = 0$ then $E_\varepsilon(u)|_{B_\varepsilon(p)} > c_0$ for some universal constant c_0.*

Hint: Work on the rescaled metric $(M, \varepsilon^{-2}g)$. There, u satisfies $\Delta u - W'(u) = 0$. Since $|u| \leq 1$ you can use Schauder estimates to obtain $C^{1,\alpha}$-bounds on u. This implies that the u must be close to zero on $B_1(p)$. Scaling back, this gives a lower bound on the potential term of the energy $\int_{B_\varepsilon(p)} W(u)/\varepsilon$.

Exercise 15 (A positive solution with Dirichlet boundary data exists if ε is small or if the region is large enough)**.** *Let ϕ be the first eigenfunction of the Laplacian on a region Ω. Show that $E_\varepsilon(\phi) < E_\varepsilon(0)$ iff $\varepsilon^2 \lambda_1 < 1 - \frac{1}{2}\frac{\int_M \phi^4}{\int_M \phi^2}$. Conclude that if $\varepsilon^2 \lambda_1 < 1/2$ then there is a positive solution to* (1) *with Dirichlet boundary data on Ω.*

Hint: By standard compactness methods (Rellich-Kondrachov) you can minimize E_ε on $W_0^{1,2}(\Omega)$. The bound $E_\varepsilon(\phi) < E_\varepsilon(0)$ guarantees that the minimizer u is not zero. Since $|\nabla|u|| = |\nabla u|$, it follows that $|u|$ is also a minimizer. Use the maximum principle to conclude that u is non-zero in Ω.

Exercise 16 (*Uniqueness of positive solutions with Dirichlet boundary data)**.** *Show that the solution from the previous exercise is unique.*

Hint: This proof from [1], *is general and it works for a wide range of semilinear equations. Let u_1 and u_2 positive solutions with Dirichlet boundary data. The* *formula $-W'(u_i) < u_i$, implies $0 < \Delta u_i + u_i$. Apply the strong maximum principle to conclude $\partial_\nu u_i < 0$ and that the functions u_1/u_2 and u_2/u_1 are in L^∞. Integrating by parts obtain the formula*

$$\int_\Omega \left(-\frac{\Delta u_1}{u_1} + \frac{\Delta u_2}{u_2}\right)(u_2^2 - u_1^2) = \int_M \left|\nabla u_1 + \frac{u_1}{u_2}\nabla u_2\right|^2 + \left|\nabla u_2 + \frac{u_2}{u_1}\nabla u_1\right|^2 \geq 0.$$

Then uniqueness, i.e. $u_1 = u_2$, follows from $\int_\Omega \left(-\frac{W'(u_1)}{u_1} + \frac{W'(u_2)}{u_2}\right)(u_2^2 - u_1^2) \geq 0$.

Remark. On $\Omega \setminus \{u = 0\}$, we can rewrite the Allen-Cahn equation as a linear equation with good decay estimates

$$\begin{aligned} 0 &= \varepsilon^2 \Delta u - W'(u) \\ &= \varepsilon^2 \Delta(u - \operatorname{sgn}(u)) - W'(u) \\ &= \varepsilon^2 \Delta(u - \operatorname{sgn}(u)) - u^3 + u \\ &= \varepsilon^2 \Delta(u - \operatorname{sgn}(u)) - u(u^2 - 1) \\ &= \varepsilon^2 \Delta(u - \operatorname{sgn}(u)) - u(u^2 - \operatorname{sgn}(u)^2) \\ &= \varepsilon^2 \Delta(u - \operatorname{sgn}(u)) - u(u + \operatorname{sgn}(u))(u - \operatorname{sgn}(u)) \\ &= \varepsilon^2 \Delta(u - \operatorname{sgn}(u)) - |u|(|u|+1)(u - \operatorname{sgn}(u)) \\ &= \varepsilon^2 \Delta v - cv, \end{aligned}$$

where $v = u - \operatorname{sgn}(u)$ and $c = |u|(|u|+1) \geq 0$. The formula $0 = \varepsilon^2 \Delta v - cv$ is a linear equation with $c > 0$. Exponential decay for solutions of this equation are standard. Indeed:

Exercise 17 (*). *Show that the function $v = u - \operatorname{sgn}(u)$ decays exponentially fast in terms of its distance to the nodal set $\{u = 0\}$. More precisely, $|v(p)| \leq Ce^{-\sigma t/\varepsilon}$, where $t = \operatorname{dist}_M(p, \{u = 0\})$.*

Hint: For the canonical 1-D solution, this follows immediately from its formula. In general, you can work on a region where the function $t = \operatorname{dist}_M(\cdot, \{u = 0\})$ is smooth. Here, you can construct supersolutions to the linear operator $\varepsilon^2\Delta - c$, where c is the function above. Using functions of the form $a(e^{-\sigma t/\varepsilon} + e^{-\sigma(t-d)/\varepsilon})$ you should be able to apply the maximum principle on a set $\{0 < t < d\}$.

Definition 6.1. We say that a function $v = v_\varepsilon$ is of order $o(\varepsilon^{\mathbb{N}})$ on a region Ω if all of its derivatives and integrals on Ω decay faster than any polynomial, i.e. $\|\nabla^k v\|_{L^p(\Omega)} = o(\varepsilon^m)$, for all $m \in \mathbb{N}$.

Remark. By Schauder, the exponential decay extends to all the derivatives of $u - \operatorname{sgn}(u)$, in particular, $|\nabla^k u| \leq Ce^{-\sigma t/\varepsilon}$, where $t = \operatorname{dist}_M(p, \{u = 0\})$. This formula implies precise estimates for $u - \operatorname{sgn}(u)$ and its derivatives on regions where $t/\varepsilon > L > 0$. Moreover, $u - \operatorname{sgn}(u)$ is $o(\varepsilon^{\mathbb{N}})$ in regions where $t/\varepsilon \to \infty$. Estimates on regions where $t/\varepsilon \leq L$, follow from blow-up arguments depending on particular geometrical assumptions. These ideas are a central aspect of the theory.

Exercise 18. *Let $B_R(0)$ be the n-dimensional ball of radius $R > 0$ in $\mathbb{R}^n$. Show that if u is the positive solution to (1) on $B_R(0)$ then u is rotationally symmetric and $1 - Ce^{-R/\varepsilon} \leq u(0) < 1$. Conclude that when $R \to \infty$ then $u \to +1$ on compacts.*

Hint: for the last claim use Schauder estimates and compactness embeddings of Holder spaces.

7. Energy control and construction of solutions by higher dimensional min-max

Since E_ε has only two isolated global minima, it is natural to expect the existence of a solution of mountain-pass type with $\mathrm{Ind}(u) \leq 1$. Moreover, by the convergence theorems from Section 9, if we can control the energy of such solutions, both from above and below, we can construct a minimal hypersurface (perhaps with a singular set of dimension at most $n-8$). This was done originally in [8] for $n \geq 3$ and extended to $n = 2$ on [10]. In this section we discuss a generalization of this construction which is presented in detail in [6].

Exercise 19. *Show that there is a Morse function $f : M \to \mathbb{R}$ such that the level sets $\Sigma_t = \{f = t\}$ move continuously with respect to the Hausdorff distance.*

Hint: the problem is that f might have local maxima (or minima). Nonetheless, these are isolated points and you can transform them into global maxima (or minima) by modifying the function just on a small neighborhood around this points.

Let f be a Morse function as in the previous exercise. As a first step on the construction of solutions, we use the one-parameter family of hypersurfaces $\Sigma_t = \{f = t\}$ to give an example of a higher-dimensional, odd family of functions $h : S^p \to W^{1,2}(M)$ with energy bounded from above independently of ε.

For each $z \in \mathbb{C}$, we define an associated distance function on M by

$$d_z : M \to \mathbb{R}_{\geq 0}, \quad d_z(x) = \mathrm{dist}_M(x, \Sigma_{\mathrm{Re}(z)}) + \mathrm{dist}_{\mathbb{C}}(z, f(M)),$$

where $f(M) = [\min_M f, \max_M f]$. For each $a = (a_0, \ldots, a_p) \in S^p$, consider the polynomial $P_a(z) = \sum_{i=0}^p a_i z^i$ and let $C(a)$ be the set of its roots in the complex plane. We then define the functions

$$d_a(x) = \begin{cases} \min\{d_z(x) : z \in C(a)\} & \text{if } C(a) \neq \emptyset \\ +\infty & \text{if } C(a) = \emptyset \end{cases}.$$

Finally, define

$$\rho_a(x) = \mathrm{sgn}_a(x) d_a(x),$$

where $\mathrm{sgn}_a(x) = \mathrm{sgn}(P_a \circ f(x))$, whenever $d_a(x) > 0$.

Exercise 20 (**). *Let ψ be the canonical one-dimensional solution. Show that $a \in S^p \to h_a = \psi \circ \rho_a \in W^{1,2}(M)$ is continuous. Moreover, $h_{-a} = -h_a$ and $\limsup_{\varepsilon \to 0} \sup_\varepsilon E_\varepsilon(h_a) = \sigma p \sup_{t \in \mathbb{R}} |\Sigma_t|$.*

Hint: once we have constructed the right distance functions d_a, the rest of the computations are long, but elementary. You can find them on [6].

The topology of $W^{1,2}(M)$ is trivial, but to hope to find critical points by means of min-max methods we need first to have some interesting topology. In this particular case, this follows from equivariant min-max methods that exploit the $\mathbb{Z}_2$-symmetries of the functional E_ε on $W^{1,2}(M)$.

Define $\mathcal{H} = W^{1,2}(M) \setminus \{0\}$ and denote by $\mathcal{S}$ the unit sphere of $\subset \mathcal{H}$. Then $\mathcal{H} \simeq \mathcal{S} \times (0, +\infty)$ is a (non-complete) smooth Hilbert manifold. We can think of the $\mathcal{S}$ as a infinite dimensional sphere. More precisely, given the natural inclusions of finite dimensional spheres $S^1 \subset S^2 \subset S^3 \subset \cdots$ the infinite dimensional sphere is defined as the set $S^\infty = \cup_{k\in\mathbb{N}} S^k$ with the largest topology making the inclusions continuous. Given an infinite set of linearly independent vectors $v_1, v_2, v_3, \ldots$ on $W^{1,2}$ (e.g. eigenvalues of the Laplacian) we can take the unit sphere on each finite dimensional span $E^k = span(v_1, \ldots, v_k)$. This gives us the inclusions $S^1 \subset S^2 \subset \cdots \subset S^\infty \subset \mathcal{H} \simeq \mathcal{S} \times (0, +\infty)$.

Now, notice that the $\mathbb{Z}_2$-action on $\mathcal{H}$ given by the antipodal map $u \to -u$ is free. Therefore, $\mathcal{H}/\mathbb{Z}_2$ is a smooth (but punctured) Hilbert manifold and we have the inclusions

$$\mathbb{RP}^1 \subset \mathbb{RP}^2 \subset \cdots \subset \mathbb{RP}^\infty \subset \mathcal{H}/\mathbb{Z}_2 \simeq \mathcal{S}/\mathbb{Z}_2 \times (0, +\infty).$$

It is a well-known fact that the cohomology of $\mathbb{RP}^\infty$ (with coefficients $\mathbb{Z}_2$) is a polynomial algebra $\mathbb{Z}_2[\gamma]$ generated by a single non-trivial element γ of the first cohomology group. It is the case that the cohomology group of $\mathcal{H}/\mathbb{Z}_2 \simeq \mathcal{S}/\mathbb{Z}_2 \times (0, +\infty)$ has the same form. This is a more subtle assertion (see [6] for the precise topological statements). However, the inclusions above should at least convince you that the cohomology of $\mathcal{H}/\mathbb{Z}_2$ is at least as complicated as $\mathbb{Z}_2[\gamma]$.)

Summarizing, all these arguments tell us that it should be possible to apply min-max methods to find critical points of E_ε in $\mathcal{H}/\mathbb{Z}_2$ (and therefore in $\mathcal{H}$) as long as we can guarantee that our min-max families stay on a bounded set and also away from the singular point which is the origin.

Given $p \in \mathbb{N}$, we define the family

$$\mathcal{F}_p = \{A \subset W^{1,2}(M) : A \text{ compact}, A = -A \text{ and } \gamma^p(A/\mathbb{Z}_2) \neq 0\}.$$

Finally, we define the p-widths

$$c_\varepsilon(p) = \inf_{A\in\mathcal{F}_p} \sup_A E_\varepsilon.$$

From the inclusions $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \mathcal{F}_2 \supset \cdots$ it follows that $c_\varepsilon(1) \leq c_\varepsilon(2) \leq \cdots$ is a monotone increasing sequence.

In [6], we use that E_ε has a Palais-Smale property on a suitable subset of $\mathcal{H}$, from classical min-max theory to conclude the following result:

Theorem 7.1. *Fix $p \in \mathbb{N}$. For ε small enough, there exists a non-trivial solution u to* (1) *with $E_\varepsilon(u) = c_\varepsilon(p) < E_\varepsilon(0)$ and* $\mathrm{Ind}(u) \leq p \leq \mathrm{Ind}(u) + \mathrm{Nul}(u)$. *Moreover,*

$$0 < \liminf_{\varepsilon\to 0} c_\varepsilon(p) \leq \limsup_{\varepsilon\to 0} c_\varepsilon(p) < +\infty.$$

Proof. Fix $p \in \mathbb{N}$. Let h be as in the previous exercise. The existence of ε, follows since $h(S^p) \subset \mathcal{F}_p$, combined with the upper bound on $E_\varepsilon(h_a)$ (which is independent of ε) and the fact that $E_\varepsilon(0) = \varepsilon^{-1}\mathrm{Vol}(M)W(0) \to \infty$. The existence of u, with $E_\varepsilon(u) = c_\varepsilon(p)$ and $\mathrm{Ind}(u) \leq p \leq \mathrm{Ind}(u)+\mathrm{Nul}(u)$ follows from the Palais-Smale condition and Ghoussoub's min-max theorem for cohomological families (see [7]). The same upper bound implies $\limsup_{\varepsilon\to 0} c_\varepsilon(p) < +\infty$. You can prove the lower bound in the following way. First, show that $c_\varepsilon(p) > 0$ (this is easy since the only solutions with energy 0 are the constants ± 1). Finally, notice that $0 < E_\varepsilon(u) < E_\varepsilon(0)$, implies that the solution is not trivial. Then, it has a nodal set from Exercise 13. Deduce a lower bound for $E_\varepsilon(u)$ from Exercise 14. □

8. Construction of solutions by minimization and gluing

Even for familiar geometries it is hard to determine how the min-max solutions constructed above will look like. Because of this, sometimes it is useful to construct solutions in other ways. In this section, we discuss how to use Exercises 15 and 16 to construct solutions by gluing methods. Some advantages of this approach is that we usually have a good picture for the nodal set as well as good bounds for their energy. On the other hand, in general it is hard to estimate the Morse index of these solutions.

Exercise 21. *Construct a solution to* (1) *on S^n with nodal set equal to an equator.*

Hint: Use Exercise 15 to solve the Dirichlet problem on each hemisphere. Choose the positive solution in one and the negative solution in the other. Use Exercise 16 to conclude that both solutions are rotationally symmetric and indeed, their gradient coincides on the equator. Conclude that the value of the solutions, their gradient and Laplacian, coincide on the equator. Deduce from this that when we glue both solutions we obtain a weak solution of (1) *(and therefore a smooth solution).*

The following exercises deal with similar ideas in different geometries.

Exercise 22. *Construct a solution to* (1) *on a rotationally symmetric torus $\mathbb{T}^n$ whose nodal set is exactly two antipodal slices of $\mathbb{T}^{n-1}$ (you can actually do it for any even number of equidistant slices).*

Exercise 23. *Construct a solution to* (1) *on S^n whose nodal set is two orthogonal equators.*

Hint: In this case the nodal set is singular. Similar ideas work for showing that after gluing we obtain a weak solution.

Exercise 24. *Construct a solution to* (1) *on $\mathbb{R}^2$ whose nodal set is two orthogonal lines.*

Hint: First, construct a solution on a ball of radius $R > 0$ to the Dirichlet problem, having nodal set two orthogonal segments of lines passing through the origin. After this, take the limit as $R \to \infty$ and use standard Schauder estimates.

Exercise 25. *Construct a solution to* (1) *on $\mathbb{R}^2$ whose nodal set is two orthogonal lines.*

Hint: First, construct a solution on a ball of radius $R > 0$ to the Dirichlet problem, having nodal set two orthogonal segments of lines passing through the origin. After this, take the limit as $R \to \infty$ and use standard Schauder estimates.

Exercise 26 (*)**.** *Construct a solution to* (1) *on S^n whose nodal set is two parallels. More precisely, $\{u = 0\} = S^n \cap \{x_{n+1} = \pm s_0\}$, for some $|s_0|< 1$.*

Hint: Define regions $A_\tau = S^n \cap \{|x_{n+1}|< \tau\}$ and $S^n \setminus A_\tau = D_\tau^+ \cup D_\tau^-$, where $D_\tau^\pm$ are the two disks forming the complement of the annulus A_τ. Glue together the positive solutions with Dirichlet condition on $D_\tau^\pm$ and the negative solution with Dirichlet condition on A_τ. Show that each solution satisfies an homogeneous Neumann condition at the boundary, which varies continuously with respect to τ. Finally, show that when ε is small enough, there exists $\tau \in [0, 1]$ such that the gradients of the solutions coincide at the boundary of A_τ. This is the solution you are looking for.

Exercise 27 (**)**.** *In the example above, you can actually prove that as $\varepsilon \to 0$, the zero level set accumulates on an equator with multiplicity 2.*

Hint: Show first that the solutions have finite energy. This can be done by expressing the solution in Fermi coordinates around the nodal set and using Exercise 17. In particular by Hutchinson-Tonegawa [9]*, the energy will accumulate on an stationary varifold. After passing to a subsequence, we can assume that the zero level set converges to either two parallels or to the equator. Prove that the energy must concentrate on this limit set (again you can use Exercise 17 for this). Since two parallels are not stationary, it follows that it must converge to the equator.*

Exercise 28. *Construct a solution to* (1) *on $\mathbb{RP}^n$ whose nodal converges to a copy of $\mathbb{RP}^{n-1}$ with multiplicity 2.*

Hint: First, show that the solutions on S^n from the previous exercise are even.

9. Connection to minimal hypersurfaces

Informally, we expect $\{u = 0\}$ to converge to a minimal hypersurface $\Sigma^{n-1} \subset M^n$ (which is perhaps singular). Moreover, $\lim_{\varepsilon\to 0} E_\varepsilon(u) = \sigma|\Sigma|$. There are elementary heuristic arguments supporting such expectations (examples can be found on [14] and `math.stanford.edu/~ryzhik/BANFF/delpino.pdf`). The following are precise versions of this fact.

Theorem 9.1 (Pacard-Ritoré [13] (see also [12, 2]))**.** *Given $\Sigma \subset M$ a smooth, separating, non-degenerate minimal hypersurface, there exists $\varepsilon_0 > 0$, such that for every $\varepsilon \in (0, \varepsilon_0)$ there is a solution u to* (1) *such that $\{u = 0\} \to \Sigma$ as a smooth graph. Moreover, $\lim_{\varepsilon\to 0} E_\varepsilon(u) = \sigma|\Sigma|$ and $\mathrm{Ind}(\Sigma) = \mathrm{Ind}(u)$.*

Theorem 9.2 (Hutchinson-Tonegawa-Wickramasekera-Guaraco [8] (see also [9, 15]))**.** *Assume there is a sequence of solutions $u = u_k$ to* (1)*, with $\varepsilon = \varepsilon_k \to 0$, such that*

$$\sup_{\varepsilon} \left(\|u\|_{L^\infty(\Omega)} + E_\varepsilon(u) + \mathrm{Ind}(u)\right) < +\infty, \tag{7}$$

then $\{u = 0\}$ converges (with respect to the Hausdorff distance) to a minimal hypersurface Σ, which is embedded outside of a set of dimension at most $n - 8$. Moreover, if $\Sigma_1, \ldots, \Sigma_k$ are the connected components of Σ, then $\lim_{\varepsilon \to 0} E_\varepsilon(u) = m_1|\Sigma_1| + \cdots + m_k|\Sigma_k|$, for $m_i \in \mathbb{N}$.

Remark. It is possible to define an index for Σ with respect to vector fields. This works even when it is not orientable. In this sense, P. Gaspar [5] showed that under hypothesis (7), one has $\mathrm{Ind}(\Sigma) \leq \lim_{\varepsilon \to 0} \mathrm{Ind}(u)$.

Theorem 9.3 (Chodosh-Mantoulidis [3])**.** *Let $n = 3$. Assume there is a sequence of solutions $u = u_k$ to (1), with $\varepsilon = \varepsilon_k \to 0$, such that*

$$\sup_{\varepsilon} \left(\|u\|_{L^\infty(\Omega)} + E_\varepsilon(u) + \mathrm{Ind}(u)\right) < +\infty,$$

then, outside of a finite set, $\{u = 0\}$ converges to a minimal hypersurface Σ as a multigraph. As above, $\lim_{\varepsilon \to 0} E_\varepsilon(u) = m_1|\Sigma_1| + \cdots + m_k|\Sigma_k|$. If $m_i \neq 1$, then Σ_i admits a positive Jacobi vector field. In particular, if Σ is non-degenerate or Ric_M is positive then $m_i = 1$, for all $i = 1, \ldots, k$.

Remark. Let Σ be a separating hypersurface. Denote by $\mathrm{Nul}(\Sigma)$ and $\mathrm{Nul}(u)$, the nullity of J and E''_ε, respectively. Chodosh-Mantoulidis [3] showed that if $m_i = 1$ for all $i = 1, \ldots, k$ then $\lim_{\varepsilon \to 0} \mathrm{Ind}(u) + \mathrm{Nul}(u) \leq \mathrm{Ind}(\Sigma) + \mathrm{Nul}(\Sigma)$.

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